A Construction of An Effective Hamiltonian from Feynman Diagrams : Application to the Light-Front Yukawa Model

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Abstract

We study a similarity transformation to construct an effective Hamiltonian systematically, which does not contain particle-number-changing interactions. We found it very advantageous that the effective Hamiltonian can be obtained from Feynman diagrams, so that we can use the covariant perturbative renormalization procedure. We apply this to the Light-Front Yukawa model as an example and confirm that the ground state does not depend on the transverse cutoff to the second order in the coupling constant.

1 Introduction

For many years, the bound state problem for QCD has been studied very well but has not come to the complete understanding until today. The main difficulty is that the low energy dynamics of QCD needs nonperturbative calculations. One of the convenient tools to solve the problem is a Hamiltonian

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approach in Light-Front (LF) field theory which is quantized on the equal LF time surface.

An important feature of the LF field theory is that the vacuum is trivial so that the Fock vacuum of the free part of the Hamiltonian is the true one. It is very useful in the relativistic bound state problem because we do not need to worry about solving the complex vacuum in contrast to the usual equal-time field theory. Even though the vacuum is trivial, to solve the Schrödinger equation for the relativistic bound state is difficult because it is natural that the eigenstates are constructed by the superposition of an infinite number of particle states allowed by symmetries of the Hamiltonian. Tamm-Dancoff (TD) approximation [1], which truncates the Fock space, that is, limits the number of particles concerned with the interaction, simplifies the practical calculations. While TD approximation was originally proposed in equal-time field theory, Perry, Harindranath, and Wilson suggested a LF version of it [2]. They pointed out that since the truncated states do not consist a complete set of the Hamiltonian, UV divergences are nonlocal and noncovariant, and counterterms which should cancel the divergences depend on the sector of the Fock space within which they act. This is called the problem of "sector-dependent counterterms".

A similarity transformation of the Hamiltonian does not change the eigenvalues and is useful for getting the effective Hamiltonian. There are two types of it. One is the transformation in momentum space, which integrates states out exchanging energy more than some energy cutoff, proposed by Głazek and Wilson[3], and independently by Wegner[4]. It is interesting that it gives the nonperturbative low energy physics and a logarithmic confining potential in LF QCD [5], although it is hard to get the effective Hamiltonian even in the lowest order in the coupling constant. Another is the transformation in the particle number space, which reduces the Hamiltonian to one which has no particle-number-changing interactions so that the transformed one can be solved easily, and was proposed by Harada and Okazaki [6]. It can avoid the problem of sector-dependent counterterms because the origin of it is that the Hamiltonian generally has particle-number-changing interactions. But its calculation is complicated and tedious. Although it was considered in the LF field theory, it is not new in the equal-time context and has been used for getting the TMO potential of nuclei [7] and called Fukuda-Sawada-Taketani's (FST's) method [8, 9]. This method gives us easier way for constructing the effective Hamiltonian and seems to be a promising way in the LF framework. However, it lacks manifest Lorentz covariance, and therefore it is difficult to tell what sort of divergences the effective Hamiltonian has before doing actual calculations. It makes the renormalization procedure more complicated than the usual covariant perturbation theory. It is highly desired to make transparent how the divergences emerge in the FST's framework.

The purpose of this paper is to prove that the effective Hamiltonian which is constructed by the FST's method can be written in terms of Feynman diagrams as the S-matrix element can be in the covariant perturbation theory, and to show that the renormalization procedure of the effective Hamiltonian is the same as the covariant one. It makes the construction of the effective Hamiltonian systematic and easier so that it allows us to perform higher order calculations [10].

This paper is organized as follows. In Sec. 2, we review the FST's method and show how the effective Hamiltonian is constructed and renormalized. In Sec. 3, as an exercise, we apply this method to the LF Yukawa model [11] and calculate the eigenvalue of the ground states of the effective Hamiltonian up to the second order in the coupling constant. In Sec. 4, we discuss the validity of our method and the results for the LF Yukawa model.

2 Formalism of similarity transformation

This section briefly reviews the FST's method partly following [9] but in a more general way. Thereafter, we show further results.

2.1 Review of the FST's method

The FST's method is to reduce the Hamiltonian to the block-diagonal form using a similarity transformation, and then we can get the effective Hamiltonian for the subspace of the Fock space without a loss of the necessary information.

We want to solve the Schrödinger equation for the second-quantized Hamiltonian

$$H|\Psi\rangle = E|\Psi\rangle, \tag{1}$$

which is consist of the free part and the interaction part:

$$H = H_0 + H', \tag{2}$$

$$H_0 : \text{free part}, \tag{3}$$

$$H' : \text{interaction part}. \tag{4}$$

To divide the space of the states explicitly, we introduce a projection operator η which satisfies

$$\eta|\psi\rangle = |\psi\rangle, \tag{5}$$

$$\eta|\text{other}\rangle = 0, \tag{6}$$

where both $|\psi\rangle$ and $|\text{other}\rangle$ are the eigenstates of H_0 and form a complete set. $|\psi\rangle$ is the target state at our disposal. For our purpose, we restrict η to the one which selects the state with definite number of particles.

Let us consider the similarity transformation that leads the Hamiltonian to the block-diagonal form in the following way

$$U^\dagger H U \, U^\dagger \begin{pmatrix} |\psi\rangle \\ |\text{other}\rangle \end{pmatrix} = \begin{pmatrix} H_{\text{eff}} & 0 \\ 0 & * \end{pmatrix} U^\dagger \begin{pmatrix} |\psi\rangle \\ |\text{other}\rangle \end{pmatrix}, \tag{7}$$

where "$*$" abbreviates the part which is not needed for our purpose. The $|\psi\rangle$ decouples from the $|\text{other}\rangle$. So we can concentrate only on the equation for $|\psi\rangle$.

We set an ansatz for the similarity (unitary) transformation operator

$$U = \begin{pmatrix} (J^\dagger J)^{-1/2} & -J^\dagger (J J^\dagger)^{-1/2} \\ J(J^\dagger J)^{-1/2} & (J J^\dagger)^{-1/2} \end{pmatrix}. \tag{8}$$

For Eq. (7), J must have the following property

$$J = \eta + (1-\eta) J \eta, \tag{9}$$

and satisfy

$$(1-\eta)\left(H' + [H_0, J] - J\langle H' J\rangle\right)\eta = 0, \tag{10}$$

where

$$\langle Q\rangle = \eta Q \eta + (1-\eta) Q (1-\eta). \tag{11}$$

From Eqs. (7), (8), and (10), the effective Hamiltonian turns out to be

$$H_{\text{eff}} = \eta \langle J^\dagger J\rangle^{-1/2} \langle J^\dagger H J\rangle \langle J^\dagger J\rangle^{-1/2} \eta. \tag{12}$$

In the interaction picture, Eq. (10) may be regarded as a differential equation

$$(1-\eta) i \frac{dJ_{\text{I}}(t)}{dt} \eta = (1-\eta)\left[H'_{\text{I}}(t) J_{\text{I}}(t) - J_{\text{I}}(t) \langle H'_{\text{I}}(t) J_{\text{I}}(t)\rangle\right]\eta. \tag{13}$$

Here, we identify $J_{\mathrm{I}}(0)$ with J. Provided we set the initial condition

$$J_{\mathrm{I}}(-\infty) = \eta, \tag{14}$$

J can be found order by order. If we expand J

$$J = \eta + \sum_{n=1}^{\infty} J_n, \tag{15}$$

where J_n is of order $(H')^n$, the resultant effective Hamiltonian is

$$\begin{aligned} H_{\mathrm{eff}} &= \frac{1}{2}\eta \biggl(H_0 + H' + H'J_1 + H'J_2 \\ &\qquad + H'J_3 + \frac{1}{4}[J_1^\dagger J_1, H'J_1] + \cdots \biggr) \eta + h.c., \end{aligned} \tag{16}$$

to the fourth order. If we assume that H' has only operator which mix $|\psi\rangle$ with $|\mathrm{other}\rangle$, Eq. (16) is

$$\begin{aligned} \langle n|H_{\mathrm{eff}}|m\rangle &= E_m \delta_{nm} \\ &+ \frac{1}{2}\langle n|H'|\alpha\rangle\langle\alpha|H'|m\rangle \left[\frac{1}{E_m - E_\alpha + i\epsilon} + \frac{1}{E_n - E_\alpha - i\epsilon} \right] \\ &- \frac{1}{2}\langle n|H'|\alpha\rangle\langle\alpha|H'|\beta\rangle\langle\beta|H'|\gamma\rangle\langle\gamma|H'|m\rangle \\ &\times \left[\frac{1}{(E_\alpha - E_m - i\epsilon)(E_\beta - E_m - i\epsilon)(E_\gamma - E_m - i\epsilon)} \right. \\ &\left. + \frac{1}{(E_\alpha - E_n + i\epsilon)(E_\beta - E_n + i\epsilon)(E_\gamma - E_n + i\epsilon)} \right] \\ &+ \frac{1}{2}\langle n|H'|\alpha\rangle\langle\alpha|H'|l\rangle\langle l|H'|\beta\rangle\langle\beta|H'|m\rangle \\ &\times \left[\frac{1}{(E_\alpha - E_m - i\epsilon)(E_\alpha - E_l - i\epsilon)(E_\beta - E_m - i\epsilon)} \right. \\ &+ \frac{1}{(E_\alpha - E_n + i\epsilon)(E_\beta - E_l + i\epsilon)(E_\beta - E_n + i\epsilon)} \\ &\left. + \frac{1}{4}\frac{2E_l - (E_n + E_m)}{(E_n - E_\alpha - i\epsilon)(E_\alpha - E_l - i\epsilon)(E_\beta - E_m - i\epsilon)(E_l - E_\beta - i\epsilon)} \right], \end{aligned} \tag{17}$$

where l, m, and n are the indices of $|\psi\rangle$, and α, β, and γ are also the indices of $|\mathrm{other}\rangle$. Sums over repeated indices are understood. E_l is the energy eigenvalue of H_0 for the state $|l\rangle$, and similarly for others. It is known that this can produce the TMO potential in the symmetrical pseudoscalar pion theory with the pseudovector coupling, which explains all the properties of the deuteron.

However, for the renormalization, it is hard to calculate the counterterms because it is not manifest what sort of the divergences emerge in the effective Hamiltonian before doing actual calculation, like Eq. (17). It is not obvious either whether it is renormalizable or not even in the renormalizable theory.

2.2 Renormalization of the effective Hamiltonian

In this subsection, we show that the effective Hamiltonian which is constructed by the FST's method is also constructed by Feynman diagrams and it can be renormalized like in the covariant perturbation theory.

We found that J can be written in the following form

$$\begin{aligned} J_1 &= -i\int_{-\infty}^{0} dt_1 (1-\eta) H'_{\mathrm{I}}(t_1)\eta, \qquad (18) \\ J_n &= (-i)^n \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n (1-\eta) H'_{\mathrm{I}}(t_1) H'_{\mathrm{I}}(t_2)\cdots H'_{\mathrm{I}}(t_n)\eta \\ &\quad - J_1 (-i)^{n-1} \int_{-\infty}^{0} dt_2 \int_{-\infty}^{t_2} dt_3 \cdots \int_{-\infty}^{t_{n-1}} dt_n \eta H'_{\mathrm{I}}(t_2) H'_{\mathrm{I}}(t_3)\cdots H'_{\mathrm{I}}(t_n)\eta \\ &\quad - J_2 (-i)^{n-2} \int_{-\infty}^{0} dt_3 \int_{-\infty}^{t_3} dt_4 \cdots \int_{-\infty}^{t_{n-1}} dt_n \eta H'_{\mathrm{I}}(t_3) H'_{\mathrm{I}}(t_4)\cdots H'_{\mathrm{I}}(t_n)\eta \\ &\quad \vdots \\ &\quad - J_{\mathrm{n-1}}(-i)\int_{-\infty}^{0} dt_n \eta H'_{\mathrm{I}}(t_n)\eta, \qquad (19) \end{aligned}$$

using mathematical induction. Define

$$\begin{aligned} G_{n+1} &= \eta\left[H' + \sum_{m=1}^{n}\frac{(-i)^m}{m!}\int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_m \right. \\ &\qquad \left. \times\, \mathrm{T}(H'_{\mathrm{I}}(0) H'_{\mathrm{I}}(t_1) H'_{\mathrm{I}}(t_2)\cdots H'_{\mathrm{I}}(t_m))\right]\eta, \qquad (20) \end{aligned}$$

$$G_n(t) = e^{iH_0 t} G_n e^{-iH_0 t}, \qquad (21)$$

and

$$F_{n+1} = \eta(H' + \sum_{m=1}^{n} H' J_m)\eta. \qquad (22)$$

From Eqs. (15), (18), and (19), F_n and G_n satisfy the following relation

$$F_{n+1} = G_{n+1} - F_{n+1}(-i)\int_{-\infty}^{0} dt\; G_{n+1}(t) + O((H')^{n+2}). \tag{23}$$

$O((H')^{n+2})$ means that we omit higher order terms than $(H')^{n+1}$. It is easy to write F_n in terms of G_n

$$F_{n+1} = G_{n+1}\left\{1 + \sum_{m=1}^{n}\left[i\int_{-\infty}^{0} dt\; G_{n+1}(t)\right]^m\right\} + O((H')^{n+2}). \tag{24}$$

Also, terms $\eta([J_1^\dagger J_1, H'J_1] + h.c.)\eta$ can be rewritten as

$$\begin{aligned} & \eta\{[J_1^\dagger J_1, H'J_1] + [J_1^\dagger H', J_1^\dagger J_1]\}\eta \\ = \;& \eta\{J_1^\dagger J_1(H' + H'J_1) - (H' + H'J_1)J_1^\dagger J_1 + h.c.\}\eta \\ = \;& -(F_2 + F_2^\dagger)G_2 + G_2(F_2 + F_2^\dagger) + h.c. \\ = \;& -[F_2 + F_2^\dagger, G_2] + h.c. \end{aligned} \tag{25}$$

From Eqs. (16), (24) and (25), we obtain the effective Hamiltonian in the useful form

$$H_{\text{eff}} = \frac{1}{2}\eta(H_0 + F_4 - \frac{1}{4}[F_2 + F_2^\dagger, G_2] + \cdots)\eta + h.c. \tag{26}$$

All interactions are written in terms only of G_n to the fourth order in H'. Since G_n is described by the time-ordered product of H' like in the well-known formula of the S-matrix operator, it allows us to use the familiar Feynman rule to construct the effective Hamiltonian. In fact, $(-i)\int_{-\infty}^{\infty} dtG_n(t)$ is the same as S-matrix to the nth order. Only the role of the time integration from $-\infty$ to ∞ is to give the total momentum conservation of the zeroth component. That is, to construct G_n, we draw the same Feynman diagrams as those which contribute to the S-matrix to the nth order and write the corresponding functions with the creation and the annihilation operators attached for the each external lines. From this construction, G_n can be renormalized perturbatively in the same way as in the covariant perturbation theory. Note that since the time integration in G_n is from $-\infty$ to 0, there are energy denominators instead of the delta functions which give energy conservation in the each vertex.

We must comment on the problem of sector-dependent counterterms. Since the renormalization procedure is the same as the covariant one, counterterms can be determined by Feynman diagrams and is independent of the state which η selects. For example, if η selects the states which consist of a pair of a quark and an anti-quark, G_n automatically includes all the one-body operators which are needed to cancel divergences which occur in the one-body sector.

Now, we have got the method of constructing the effective Hamiltonian which has no particle-number-changing interactions, and proved that the necessary renormalization procedure is the same as the familiar covariant perturbation theory, so we apply this method to the LF Yukawa model and make sure that the effective Hamiltonian is not divergent.

3 Application to the LF Yukawa model

Let us apply our method to the LF Yukawa model. We calculate the ground state energy numerically to the second order in the coupling constant. Of course, the calculations to this order are not so different from the other methods [12]; the advantage of the present formulation becomes apparent in the higher orders [10]. The reason why we present the second order calculation here is to demonstrate some features of our method and to clarify what would be expected in the next order.

The Lagrangian of the this model is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \sum_i \bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i + ig\gamma_5\phi)\psi_i. \tag{27}$$

Hereafter, we call ψ_i quark and ϕ pion. Using LF quantization with Dirac quantization method, dynamical fields must satisfy the following commutation relations

$$[\,\phi(x), \partial_-\phi(y)\,]_{x^+=y^+} = \frac{i}{2}\delta(x^- - y^-)\delta^2(x_\perp - y_\perp), \tag{28}$$

$$\{\xi_i(x), \xi_j^\dagger(y)\}_{x^+=y^+} = \frac{1}{\sqrt{2}}\delta(x^- - y^-)\delta^2(x_\perp - y_\perp)\delta_{ij}\Lambda_+, \tag{29}$$

where $\xi_i(x)$ are the dynamical part of the Dirac fields

$$\xi_i(x) = \Lambda_+\psi_i(x). \tag{30}$$

$\Lambda_\pm$ are the projection operators of the Dirac fields and defined as

$$\Lambda_\pm = \frac{1}{\sqrt{2}}\gamma^0\gamma^\pm, \tag{31}$$

$$\Lambda_\pm\Lambda_\pm = \Lambda_\pm, \quad \Lambda_\pm\Lambda_\mp = 0, \quad \Lambda_+ + \Lambda_- = 1. \tag{32}$$

From Eq. (27), the familiar Legendre transformation gives the LF Hamiltonian

$$\begin{aligned} P^- &= \int dx^+ d^2x_\perp \left\{ \frac{1}{2}(\partial_\perp\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \sum_i \xi_i^\dagger \frac{-\partial_\perp^2 + m_i^2}{\sqrt{2}i\partial_-}\xi_i \right. \\ &\quad -ig\sum_i \left[\xi_i^\dagger \gamma^0\gamma_5\phi \frac{1}{\sqrt{2}i\partial_-}(i\alpha_\perp\cdot\partial_\perp + m_i\beta)\xi_i - h.c.\right] \\ &\quad \left. +\frac{g^2}{2}\sum_i \left[(\xi_i^\dagger\phi)\frac{1}{\sqrt{2}i\partial_-}(\phi\xi_i) - \frac{1}{\sqrt{2}i\partial_-}(\phi\xi_i^T)(\xi_i^*\phi)\right]\right\}, \end{aligned} \tag{33}$$

$$\alpha_\perp = \gamma^0\gamma_\perp, \quad \beta = \gamma^0. \tag{34}$$

In the interaction picture, the LF Hamiltonian is

$$\begin{aligned} P_{\mathrm{I}}^-(x^+) &= \int dx^- d^2x_\perp \left\{ \frac{1}{2}(\partial_\perp\phi_{\mathrm{I}})^2 + \frac{1}{2}\mu^2\phi_{\mathrm{I}}^2 + \sum_i \bar{\psi}_{\mathrm{I}i}\gamma^+ \frac{-\partial_\perp^2 + m_i^2}{2i\partial_-}\psi_{\mathrm{I}i} \right. \\ &\quad -ig\sum_i \bar{\psi}_{\mathrm{I}i}\gamma_5\phi_{\mathrm{I}}\psi_{\mathrm{I}i} \\ &\quad \left. +\frac{g^2}{2}\sum_i \left[\bar{\psi}_{\mathrm{I}i}\phi_{\mathrm{I}}\frac{\gamma^+}{2i\partial_-}(\phi_{\mathrm{I}}\psi_{\mathrm{I}i}) - (\frac{\gamma^+}{2i\partial_-}\phi_{\mathrm{I}}\psi_{\mathrm{I}i})^{\mathrm{T}}(\bar{\psi}_{\mathrm{I}i}\phi_{\mathrm{I}})^{\mathrm{T}}\right]\right\} \\ &\quad +P_{\mathrm{CT}}^-(x^+), \end{aligned} \tag{35}$$

where $\psi_{\mathrm{I}i}(x)$ are new fields defined as

$$\psi_{\mathrm{I}i}(x) = \left[1 + \frac{1}{\sqrt{2}i\partial_-}(i\alpha_\perp\cdot\partial_\perp + m_i\beta)\right]\xi_{\mathrm{I}i}(x), \tag{36}$$

and satisfy the free Dirac equation. P_{CT}^- are counterterms, which emerge by shifting μ, m_i and g.

Applying our method, the LF effective Hamiltonian is

$$\begin{aligned} P_{\mathrm{eff}}^- &= \frac{1}{2}\Big\{ \eta\, P_{\mathrm{I}}^-(0)\,\eta - i\int_{-\infty}^0 dx^+\, \eta\, \mathrm{T}^+(P_{\mathrm{I}}^{\prime -}(0)P_{\mathrm{I}}^{\prime -}(x^+))\,\eta \\ &\quad +O(g^4)\Big\} + h.c., \end{aligned} \tag{37}$$

to g^2 order. In the LF coordinates, we must use LF-time-ordered product T^+ instead of usual time-ordered product T. If we assume that LF amplitudes are equivalent to the covariant one except for the vacuum, it is easy to estimate Eq. (37) by Feynman diagrams.

Here, we define η as the one which selects only the two-body $q\overline{q}$ state. Since possible graphs are the same as those needed in constructing the S-matrix, we can immediately imagine these which contribute to Eq. (37). The Feynman diagrams associated with the effective Hamiltonian to the second order are shown in Fig. 1. For the merit of the LF field theory, there is no disconnected vacuum graphs.

Terms which should be renormalized are only quark self-energy terms in this order. These for quarks correspond to the first and the third graphs in the second line of Fig. 1 and are written as

$$
\begin{aligned}
& \frac{1}{2}\sum_i\sum_\lambda\int_{p_1}\int_{p_2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4} \\
& \quad\times\left[\frac{1}{p_1^- - q^- - k^- - i\epsilon}+\frac{1}{q^- + k^- - p_2^- - i\epsilon}\right] \\
& \quad\times(2\pi)^3\delta(p_1^+ - q^+ - k^+)\delta^2(p_{1\perp} - q_\perp - k_\perp) \\
& \quad\times(2\pi)^3\delta(q^+ + k^+ - p_2^+)\delta^2(q_\perp + k_\perp - p_{2\perp}) \\
& \quad\times\bar{u}_i(p_1,\lambda)\left[\frac{i}{q^2-\mu^2+i\epsilon}(-g\gamma_5)\frac{i(\not{k}+m_i)}{k^2-m_i^2+i\epsilon}(-g\gamma_5)\right. \\
& \qquad\qquad\left. -im_i^{(2)}(2\pi)^4\delta^4(q-k)\right]u_i(p_2,\lambda)\; b_{i\lambda}^\dagger(p_1)b_{i\lambda}(p_2) \\
= & \frac{1}{2}\sum_i\sum_\lambda\int_p\frac{1}{2p^+}\bar{u}_i(p,\lambda)\Sigma_i^{(2)}(p)u_i(p,\lambda)\; b_{i\lambda}^\dagger(p)b_{i\lambda}(p), \qquad (38)
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_i^{(2)}(p) &= i\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-\mu^2+i\epsilon}(-g\gamma_5)\frac{i(\not{p}-\not{q}+m_i)}{(p-q)^2-m_i^2+i\epsilon}(-g\gamma_5) \\
&\quad + \delta m_i^{(2)}, \qquad (39)
\end{aligned}
$$

$$
\int_p = \int\frac{dp^+d^2p_\perp}{2p^+(2\pi)^3}\theta(p^+), \qquad (40)
$$

$$
p^2 = p_1^2 = p_2^2 = m_i^2, \qquad (41)
$$

and $\delta m_i^{(2)}$'s are the mass renormalization constants of order g^2, which emerge by the shift of the quark masses m_i. In the construction of these terms, we use the LF energy p^- denominators for the each vertex instead of the δ-functions of the p^- conservation because the LF time integration is from $-\infty$ to 0. However, from $p_1^+ = p_2^+$, $p_{1\perp} = p_{2\perp}$ and Eq. (41), $p_1^- = p_2^-$ is satisfied, therefore

$$\left[\frac{1}{p_1^- - q^- - k^- - i\epsilon} + \frac{1}{q^- + k^- - p_2^- - i\epsilon}\right] = 2\pi i \delta(p_1^- - q^- - k^-). \tag{42}$$

Obviously, $\Sigma_i^{(2)}(p)$ is the same as quark self-energy terms in the covariant perturbation theory. It is convenient to impose the physical renormalization condition

$$\bar{u}_i(p,\lambda)\Sigma_i^{(2)}(p)u_i(p,\lambda) = 0 \quad (p^2 = m_i^2), \tag{43}$$

so that the quark masses become the physical one. For anti-particle, the same argument is applied.

The rest are the pion exchange part and the quark annihilation part which are represented by the last two graphs in Fig. 1. These are

$$\begin{aligned}
&\frac{g^2}{2}\sum_i\sum_j\prod_{n=1}^{4}\left[\sum_{\lambda_n}\int_{p_n}\right](2\pi)^3\delta^3(p_1 - p_2 - p_3 + p_4) \\
&\quad\times \bar{u}_i(p_1,\lambda_1)\gamma_5 u_i(p_2,\lambda_2)\bar{v}_j(p_3,\lambda_3)\gamma_5 v_j(p_4,\lambda_4) \\
&\quad\times\left[\frac{1}{2(p_1^+ - p_2^+)(p_1^- - p_2^-) - (p_{1\perp} - p_{2\perp})^2 + \mu^2}\right. \\
&\qquad\left. + \frac{1}{2(p_3^+ - p_4^+)(p_3^- - p_4^-) - (p_{3\perp} - p_{4\perp})^2 + \mu^2}\right] \\
&\quad\times \eta b^\dagger_{i\lambda_1}(p_1)d^\dagger_{j\lambda_3}(p_3)d_{j\lambda_4}(p_4)b_{i\lambda_2}(p_2)\eta \\
&-\frac{g^2}{2}\sum_i\prod_{n=1}^{4}\left[\sum_{\lambda_n}\int_{p_n}\right](2\pi)^3\delta^3(p_1 + p_2 - p_3 - p_4) \\
&\quad\times \bar{u}_i(p_1,\lambda_1)\gamma_5 v_i(p_2,\lambda_2)\bar{v}_i(p_3,\lambda_3)\gamma_5 u_i(p_4,\lambda_4) \\
&\quad\times\left[\frac{1}{2(p_1^+ + p_2^+)(p_1^- + p_2^-) - (p_{1\perp} + p_{2\perp})^2 + \mu^2}\right. \\
&\qquad\left. + \frac{1}{2(p_3^+ + p_4^+)(p_3^- + p_4^-) - (p_{3\perp} + p_{4\perp})^2 + \mu^2}\right] \\
&\quad\times \eta b^\dagger_{i\lambda_1}(p_1)d^\dagger_{i\lambda_2}(p_2)d_{i\lambda_3}(p_3)b_{i\lambda_4}(p_4)\eta,
\end{aligned} \tag{44}$$

which are all the second order interactions in the LF effective Hamiltonian. Of course, since it does not have the particle-number-changing interactions, the eigenstate must consist only of the two-body $q\overline{q}$ states.

We set the total transverse momentum $P_\perp$ to 0 for simplicity. The eigenstate is

$$
\begin{aligned}
|\Psi_{ij}(P^+,m)\rangle \;=\; & \sum_{\lambda_1}\sum_{\lambda_2}\int_0^1 dx \int_0^\infty d\kappa \int_0^{2\pi} d\varphi \frac{1}{2(2\pi)^3}\sqrt{\frac{\kappa}{x(1-x)}} \\
& \times e^{i(m-\lambda_1/2-\lambda_2/2)\varphi}\Psi_{ij}(x,\kappa;\lambda_1,\lambda_2,m) b^\dagger_{i\lambda_1}(p_1) d^\dagger_{j\lambda_2}(p_2)|0\rangle, \quad (45)
\end{aligned}
$$

where

$$
\begin{aligned}
& p_1^+ = xP^+,\ p_2^+ = (1-x)P^+,\ (p_{1\perp}-p_{2\perp})/2 = \kappa_\perp, \\
& \kappa = |\kappa_\perp|,\ \tan\varphi = \tfrac{\kappa_\perp^2}{\kappa_\perp^1}.
\end{aligned} \quad (46)
$$

m is the eigenvalue of the third component of the total angular momentum J_3. In the LF coordinates, the squared total angular momentum J^2 is not a good quantum number, but the third component J_3, the helicity, is.

We discretized x with L equally-spaced points, in the numerical calculations of diagonalization. We put a transverse cutoff $\Lambda_\perp$ for κ and, also used the following variable

$$
z = \left(\frac{\kappa}{\Lambda_\perp}\right)^{1/3}, \quad (47)
$$

instead of κ and discretized z with N equally-spaced points because the wavefunctions are sharp for $\kappa \sim 0$, but flat for $\kappa \sim \Lambda_\perp$.

The results are shown in Fig. 2 and Fig. 3. We set the parameters $L = 10$, $N = 30$, $m = 0$, the quark mass $m_1 = 1.0$ GeV, the anti-quark mass $m_2 = 1.0$ GeV and the pion mass $\mu = 0.01$ GeV in all the cases. We define $\alpha_g = g^2/4\pi$. Fig. 2 shows α_g dependence of the binding energy of the ground state for various transverse cutoff $\Lambda_\perp$. (a) is the case that the quark flavor is different from the anti-quark one, that is, excluding the quark annihilation part. (b) is the case of including it. In (a), the binding energy is almost independent of $\Lambda_\perp$ in $\alpha_g < 1.5$, but it is slightly negative so that the ground state is not a bound state. Even though the binding energy becomes larger as α_g grows, the ground state is not bound because it apparently depends on $\Lambda_\perp$ for large α_g. It should not be surprising that the results for $\alpha_g > 1.5$ is not reliable because our formulation is based on the perturbation theory. (b) has more strong dependence on $\Lambda_\perp$ than (a). The reason may be understood from Fig. 3, which shows $\Lambda_\perp$ dependence of the binding energy of the ground state for

$g = 3$. If the quark annihilation part is included, the LF effective Hamiltonian needs the renormalization of the pion mass because $\Lambda_\perp$ dependence seems to be quadratic and it corresponds to a quark loop. But the pion mass counterterm does not appear until the fourth order, so to eliminate the strong $\Lambda_\perp$ dependence, we must calculate the LF effective Hamiltonian in the next order in α_g.

4 Summary and Discussion

In this paper, we showed that the effective Hamiltonian, which is made by the similarity transformation in the particle number space, can be renormalized systematically to the fourth order in H'. Since the effective Hamiltonian can be written in terms of only G_n which has the same form of the formula of the S-matrix operator, we can use the Feynman rule for constructing the effective Hamiltonian. Using this method, the construction of the effective Hamiltonian become easier than the traditional one especially in the higher order. It can be also applied not only to LF field theory but also to the ordinary equal-time one.

We proved here that the effective Hamiltonian can be written in terms of the time-ordered products to the forth order by the explicit calculation. But the general proof for higher orders is lacking. We do not know how the general proof goes, we think that it is likely that this feature persits to all orders.

In the Yukawa model, although we did not get bound states, we showed that the lowest-energy state does not depend on the transverse cutoff $\Lambda_\perp$ for the coupling constant α_g less than 1.5. We also found that the pion mass should be renormalized in next order to cancel the divergence in the pion self-energy once the quark annihilation part included. We are now extending the present work to the next order to confirm the independence of the transverse cutoff [10].

It is interesting to note that this method might also be applied to the similarity transformation in momentum space because Eq. (26) does not depend on the kinds of η. If so, to get the effective Hamiltonian would be much easier than the traditional way.

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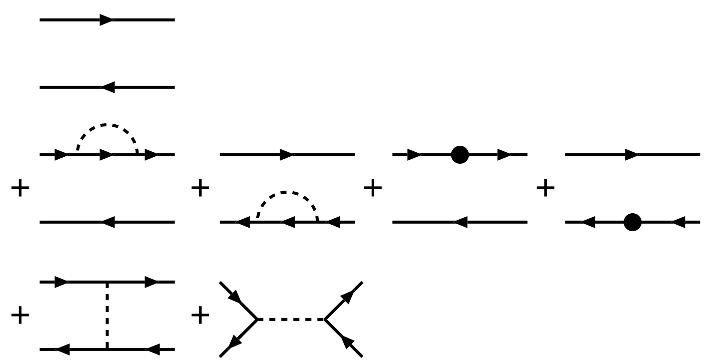
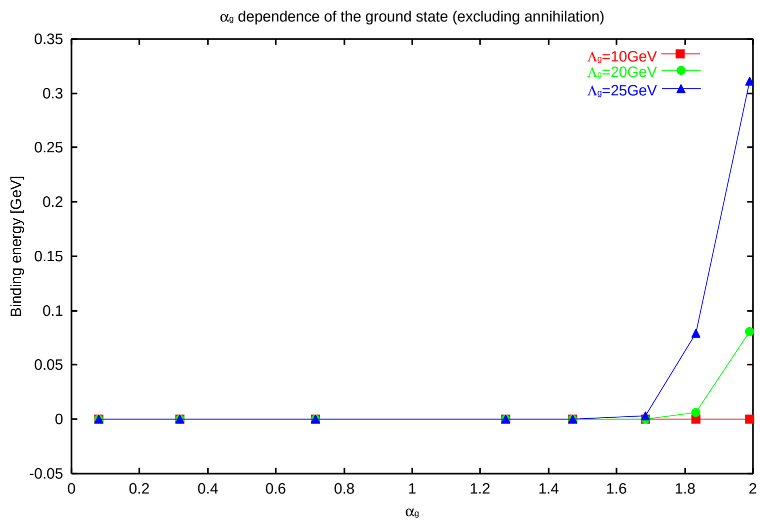
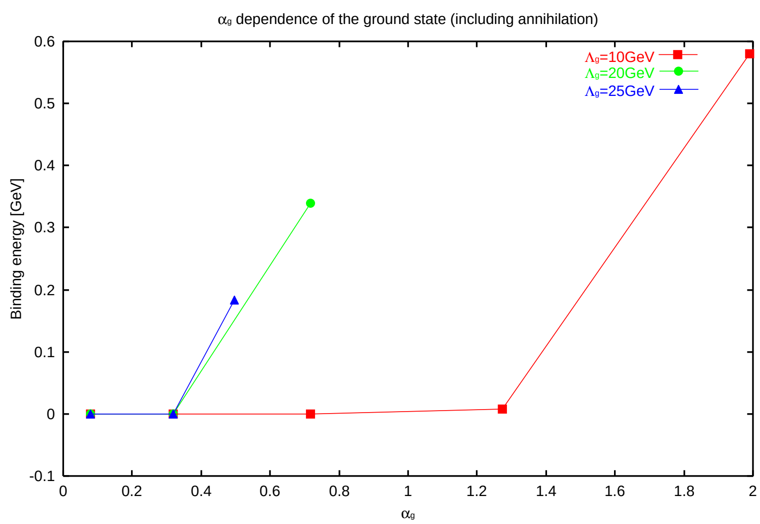


Figure 1: Feynman diagrams which are necessary to construct the effective Hamiltonian to the second order in g. The first diagram is the free part. The next four diagrams are self-energy parts and the blobs mean the mass counterterms. The last two diagrams corresponds to the pion exchange part and the quark annihilation part.



(a)



(b)

Figure 2: α_g dependence of the binding energy of the ground state for various $\Lambda_\perp$. α_g is defined as $g^2/4\pi$. The calculations are done for $L = 10$, $N = 30$, $m = 0$ GeV, $m_1 = m_2 = 1.0$ GeV, and $\mu = 0.01$ GeV. (a) is the case of excluding the quark annihilation part and (b) is the case of including it.

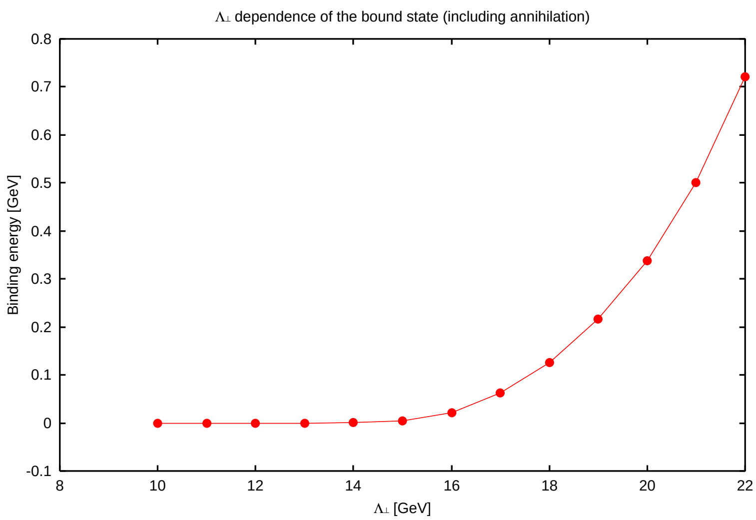


Figure 3: $\Lambda_\perp$ dependence of the binding energy of the ground state including the quark annihilation part. The calculation done for $L = 10$, $N = 30$, $\alpha_g = 0.716$, $m = 0$, $m_1 = m_2 = 1.0$ GeV, and $\mu = 0.01$ GeV.